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A TOPOLOGICAL MINIMAX INEQUALITY WITH γ-DCQCV AND ITS APPLICATIONS

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Abstract. In this paper, using the γ-diagonally $\mathcal{C}$-quasiconcave condition, we will prove a new minimax inequality in a non-convex subset of a topological space which generalizes Fan's minimax inequality and its generalizations in several aspects.

1. Introduction

Fan's minimax inequality [3] is well-known and has become a versatile tool in nonlinear and convex analysis, and there have been numerous generalizations of Fan's minimax inequality using general concave conditions as in [4-9]. However, a number of generalizations of Fan's minimax inequality and their applications always work only in topological vector spaces, i.e., the linear structure and the continuity are always needed (e.g., see [5,6,8,9]).

In a recent paper [4], the author proved a generalization of Fan's minimax inequality using γ-diagonally $\mathcal{C}$-quasiconcave (simply, γ-DCQCV) condition, which means his minimax inequality works in a topological space without assuming the linear structure.

In this paper, using the γ-DCQCV condition, we will prove a new minimax inequality in a non-convex subset of a topological space which generalizes Fan's minimax inequality and its generalizations in several aspects. As an application, we will prove a basic inequality which is an useful tool for applications in a topological space.

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2. Preliminaries

We begin with some notations and definitions. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. Denote by $[0,1]^n$ the Cartesian product of n unit intervals $[0,1]\times\cdots\times[0,1]$, and denote the unit simplex in $[0,1]^n$ by Δ_n, and simply denote $\lambda = (\lambda_1,\ldots,\lambda_n)\in\Delta_n$.

In a recent paper [4], the author introduced a general concave condition in a topological space without the linear structure as follows:

Definition 2.1. Let Y be a nonempty subset of a topological space X. Then $f : X\times Y\to\mathbb{R}\cup\{+\infty\}$ is called *diagonally $\mathcal{C}$-quasiconcave* (simply, DCQCV) *on Y* if for every $n\ge 2$, whenever n points $y_1,\ldots,y_n\in Y$ are given, there exists a continuous function $\phi_n : \Delta_n\to Y$ such that

$$f(\phi_n(\lambda),\phi_n(\lambda)) \ge \min\{f(\phi_n(\lambda_1,\ldots,\lambda_n),y_i) \mid i\in J\}$$

for all $\lambda=(\lambda_1,\ldots,\lambda_n)\in\Delta_n$, where $J=\{i\in\{1,2,\cdots,n\}\mid\lambda_i\neq 0\}$; and f is called *γ-diagonally $\mathcal{C}$-quasiconcave* (simply, γ-DCQCV) *on Y for some $\gamma\in(-\infty,\infty]$* if for every $n\ge 2$, whenever n points $y_1,\ldots,y_n\in Y$ are given, there exists a continuous function $\phi_n : \Delta_n\to Y$ such that

$$\gamma \ge \min\{f(\phi_n(\lambda_1,\ldots,\lambda_n),y_i)\mid i\in J\}$$

for all $\lambda=(\lambda_1,\ldots,\lambda_n)\in\Delta_n$, where $J=\{i\in\{1,2,\cdots,n\}\mid\lambda_i\neq 0\}$. Similarly, we can define the diagonally $\mathcal{C}$-quasiconvex (simply, DCQCX) and γ-diagonally $\mathcal{C}$-quasiconvex (simply, γ-DCQCX) conditions for f.

Remark 2.2. (1) As remarked in [4], in Definition 2.1, when X is a topological vector space, the diagonal quasiconcavity and γ-diagonal quasiconcavity, introduced by Chang-Zhang [5] and Zhou-Chen [6], clearly imply the DCQCV and γ-DCQCV conditions for f, respectively, by letting $\phi_n(\lambda_1,\ldots,\lambda_n) := \lambda_1x_1+\cdots+\lambda_nx_n$ for all $\lambda=(\lambda_1,\ldots,\lambda_n)\in\Delta_n$, and $x_1,\ldots,x_n\in X$. Therefore, the γ-DCQCV condition generalizes many previous concave conditions including the quasiconcave, CF-concave, $\mathcal{C}$-concave, diagonally quasiconcave, and γ-diagonally quasiconcave conditions without assuming the linear structure.
(2) When X is a topological vector space and the case $X=Y$ in the above definition, as shown in [4], we can obtain the following implication diagram for general concave conditions, and the converse can not be true in each step:

linear $\Longrightarrow$ concave $\Longrightarrow$ quasiconcave $\Longrightarrow$ CF-concave $\Longrightarrow$ $\mathcal{C}$-concave

$\Longrightarrow$ diagonally $\mathcal{C}$-concave $\Longrightarrow$ diagonally $\mathcal{C}$-quasiconcave

$\Longrightarrow$ γ-DCQCV

Throughout this paper, all topological spaces are assumed to be Hausdorff, and for the other standard notations and terminologies, we shall refer to [4-7].

3. A non-convex minimax inequality and its applications

Using the γ-DCQCV condition, we will prove a new non-convex minimax inequality in a topological space without assuming the linear structure which is slightly different from Theorem 1 in [4] as follow:

THEOREM 3.1. *Let X be a compact topological space, D a nonempty subset of X having more than two points, and let $f, g : X \times X \to \mathbb{R} \cup \{+\infty\}$ be two functions satisfying*

(1) for each $(x,y) \in X \times X$, $f(x,y) \le g(x,y)$;

(2) for each $y \in X$, $x \mapsto f(x,y)$ is lower semicontinuous on X;

(3) for each $x \in X$, $y \mapsto g(x,y)$ is γ-DCQCV on D when $\gamma := \sup_{x\in X} g(x,x)$;

(4) for each $y \in X \setminus D$, $f(x,y) \le \gamma$ for all $x \in D$.

Then the minimax inequality

$$\min_{x\in X} \sup_{y\in X} f(x,y) \le \sup_{x\in X} g(x,x)$$

holds.

Proof. If $\gamma = +\infty$, then we have done so that we may assume that $\gamma < +\infty$. Suppose the contrary, i.e.,

$$\min_{x\in X} \sup_{y\in X} f(x,y) > \sup_{x\in X} g(x,x) = \gamma.$$

Then, for each $x \in X$, there exists an $y \in X$ such that $f(x,y) > \gamma$.

For each $y \in X$, we let

$$U(y) := \{x \in X \mid f(x,y) > \gamma\}.$$

Note that $y \notin U(y)$ since $f(y,y) \le g(y,y) \le \gamma$ for all $y \in X$. By the assumption (2), $U(y)$ is (possibly empty) open in X for each $y \in X$, and $X = \bigcup_{y\in X} U(y)$. Since X is compact, there exists a finite set

$\{y_1, \ldots, y_n\} \subseteq X$ such that $X = \bigcup_{i=1}^{n} U(y_i)$. By the coercive assumption (4), we know that for each $y \in X \setminus D$, $U(y) \subset X \setminus D$ so that $\bigcup_{y \in X \setminus D} U(y) \subseteq X \setminus D$ and so $D \subseteq \bigcup_{y \in D} U(y)$. Therefore, by renumbering the indices, we can divide the finite set $\{y_1, \ldots, y_n\}$ into two subsets $\{y_1, \ldots, y_m\} \subset D$ and $\{y_{m+1}, \ldots, y_n\} \subset X \setminus D$ such that $D \subseteq \bigcup_{i=1}^{m} U(y_i)$ and $X = \bigcup_{i=1}^{n} U(y_i)$. Here we note that $\{y_{m+1}, \ldots, y_n\}$ might be an empty set.

Since X is compact, there exists a partition of unity $\{\alpha_1, \cdots, \alpha_n\}$ subordinated to the open covering $\{U(y_1), \ldots, U(y_n)\}$ of X, i.e.,

$$0 \leq \alpha_i(x) \leq 1, \quad \sum_{i=1}^{n} \alpha_i(x) = 1 \text{ for all } x \in X, \ i = 1, \ldots, n;$$
$$\text{and if } x \notin U(y_j) \text{ for some } j, \text{ then } \alpha_j(x) = 0.$$

For the nonempty finite set $\{y_1, \ldots, y_m\} \subset D$, since g is γ-DCQCV on D, there exists a continuous mapping $\phi_m : \Delta_m \to D$ satisfying the condition

$$\gamma = \sup_{x \in X} g(x,x) \geq \min\{g\big(\phi_m(\lambda_1, \ldots, \lambda_m), y_j\big) \mid j \in J\} \tag{$*$}$$

for all $(\lambda_1, \ldots, \lambda_m) \in \Delta_m$, where $J = \{j \in \{1, 2, \cdots, m\} \mid \lambda_j \neq 0\}$.

For each $x \in D\big(\subseteq \bigcup_{i=1}^{m} U(y_i)\big)$, we have $\alpha_j(x) = 0$ for all $j = m+1, \ldots, n$ so that $\sum_{i=1}^{m} \alpha_i(x) = 1$. Therefore, we can define a continuous map $\alpha : D \to \Delta_m$ by $\alpha(x) := (\alpha_1(x), \cdots, \alpha_m(x))$ for each $x \in D$, and consider a continuous map $\Phi : \Delta_m \to \Delta_m$ defined by

$$\Phi(\lambda) := \alpha \circ \phi_m(\lambda) = \Big(\alpha_1(\phi_m(\lambda)), \ldots, \alpha_m(\phi_m(\lambda))\Big) \quad \text{for each } \lambda \in \Delta_m.$$

Since ϕ_m and each α_i are continuous, $\Phi : \Delta_m \to \Delta_m$ is continuous on a compact convex set Δ_m. Therefore, by Brouwer's fixed point theorem, there exists a fixed point $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m) \in \Delta_m$ for Φ, i.e.,

$$\Phi(\bar{\lambda}) = \alpha \circ \phi_m(\bar{\lambda}) = \Big(\alpha_1(\phi_m(\bar{\lambda})), \ldots, \alpha_m(\phi_m(\bar{\lambda}))\Big) = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m).$$

Here we let $\bar{x} = \phi_m(\bar{\lambda})$, then $\bar{\lambda} = \alpha(\bar{x})$. By the inequality $(*)$, we have

$$\gamma = \sup_{x \in X} g(x,x) \geq \min\{g(\bar{x}, y_j) \,|\, j \in J\} \geq \min\{f(\bar{x}, y_j) \,|\, j \in J\}, \tag{$\dagger$}$$

where $J = \{j \in \{1, 2, \cdots, m\} \mid \bar{\lambda}_j \neq 0\}$. Here we note that for each $j \in J$, $\bar{\lambda}_j \neq 0$ which means $\alpha_j(\bar{x}) \neq 0$. Hence, we have that for each $j \in J$, $\bar{x} \in U(y_j)$ so that $f(\bar{x}, y_j) > \gamma$. Therefore, from the inequality (†), we have

$$\gamma \geq \min\{g(\bar{x}, y_j) \mid j \in J\} \geq \min\{f(\bar{x}, y_j) \mid j \in J\} > \gamma,$$

which is a contradiction. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes Fan's minimax inequality [3], minimax inequalities due to Chang-Zhang [5], Kim-Lee [7], Kim [4], Tan [8], and Zhou-Chen [6] in the following aspects:

(a) the function $y \mapsto f(x, y)$ need not be quasconcave, $\mathcal{C}$-concave, γ-DQCV nor diagonally $\mathcal{C}$-concave on X. Indeed, the weaker assumption γ-DCQCV is sufficient.

(b) When $X = D$ in Theorem 3.1, the coercivity assumption (4) is automatically satisfied so that Theorem 3.1 reduces to Theorem 1 [4]. Indeed, in Theorem 1 [4], γ-DCQCV condition should be needed on a compact set D, however, in the above, γ-DCQCV condition is needed on any nonempty subset D of X having more than two points.

Next, we can obtain a generalization of Fan's minimax inequality in a topological vector space as an application of Theorem 3.1.

Theorem 3.3. *Let X be a compact convex subset of a topological vector space, D a nonempty subset of X having more than two points, and let $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a function satisfying*

(1) for each $x \in X$, $y \mapsto f(x, y)$ is quasconcave on X;

(2) for each $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous on X;

(3) for each $y \in X \setminus D$, $f(x, y) \leq \sup_{z \in X} f(z, z) = \gamma$ for all $x \in D$;

(4) for each $y \in D$, $f(x, y) > \gamma$ for all $x \in X \setminus D$.

Then we have $\min_{x \in X} \sup_{y \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$.

Proof. If $\gamma = +\infty$, then we have done so that we may assume that $\gamma < +\infty$. In order to apply Theorem 3.1, it suffices to show that for each $x \in X$, $y \mapsto f(x, y)$ is γ-diagonally $\mathcal{C}$-quasiconcave on D. Indeed, in the previous Remark 3.2, we know that quasiconcave condition implies the γ-DCQCV condition. Here, for the completeness, we can give a direct proof by using the well-known Fan minimax inequality. For every $n \geq 2$, let n points $\{y_1, \ldots, y_n\} \subset D$ be given and let $K := co\{y_1, \ldots, y_n\}$ be a compact convex subset of X. Consider the restriction $g = f|_K : K \times K \to \mathbb{R}$ of f on $K \times K$. Then g satisfies the whole assumption of Theorem 1 due to Fan [3] so that we have

$$\min_{x \in K} \sup_{y \in K} g(x, y) \leq \sup_{y \in K} g(y, y);$$

i.e., there exists $x_0 \in K$ such that

$$\sup_{y\in K} g(x_0, y) \leq \sup_{y\in K} g(y, y) \leq \sup_{y\in X} f(y, y) = \gamma.$$

Therefore, by the assumption (4), $x_0 \in K$ must be an element of D since $K \cap D \neq \emptyset$.

We now define a constant continuous map $\phi_n : \Delta_n \to D$ by $\phi_n(\lambda) := x_0$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$. Then, we have

$$\min\{f(\phi_n(\lambda), y_i) \mid i \in J\} \leq \sup\{f(x_0, y_i) \mid i \in J\} \leq \sup_{y\in K} f(y, y) \leq \gamma,$$

where $J = \{j \in \{1, 2, \cdots, n\} \mid \lambda_j \neq 0\}$, so that f is γ-diagonally $\mathcal{C}$-quasiconcave on D which completes the proof. □

Remark 3.4. (1) Modifying the proof of Theorem 3.3, we can replace the quasiconcave assumption (1) by the following general concave conditions without affecting the conclusion:

(a) the linear or concave condition due to von Neumann [10];
(b) the quasiconcave condition due to Fan [3];
(c) the $\mathcal{C}$-concave condition due to Kim-Lee [7];
(d) the diagonally concave condition due to Zhou-Chen [6];

(2) When $X = D$ is compact and convex, then the assumptions (3) and (4) of Theorem 3.3 are automatically satisfied so that the minimax inequality due to Fan [3] is a consequence of Theorem 3.3.

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